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Some Traffic Overflow Problems with a Large Secondary Queue

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When calls offered to a primary group of trunks find all of them busy, provisions are often made for these calls to overflow to other groups of trunks. Such traffic overflow systems have been of interest for a long time, but recently overflow systems that allow for some calls to be queued have been of importance. In this paper we analyze a traffic overflow system with queuing. The system consists of two groups, a primary and a secondary. We consider two cases which differ in the treatment of demands waiting in the primary queue. We present an analytical approach which is suitable if the secondary queue is large, or even infinite, and we contrast it with an earlier approach of ours which is more suitable if the secondary queue is not large. The analysis considerably reduces the dimensions of the problem and simplifies the calculation of various steady-state quantities of interest. Our results include expressions for the loss probabilities, the average waiting times in the queues, and the average number of demands in service in each group.

I. INTRODUCTION

In an earlier paper,[1] we analyzed a traffic overflow system with queuing. The system consists of two groups, a primary and a secondary, with n_k servers and q_k waiting spaces, which receive demands from independent Poisson sources S_k with arrival rates $\lambda_k > 0$, $k = 1$ and 2, respectively, as depicted in Fig. 1. The service times of the demands are independent and exponentially distributed with mean service rate $\mu > 0$. If all n_2 servers in the secondary are busy when a demand from S_2 arrives, the demand is queued if one of the q_2 waiting spaces is available; otherwise, it is lost (blocked and cleared from the system). Demands waiting in the secondary queue enter service (in some prescribed order) as servers in the secondary become free.

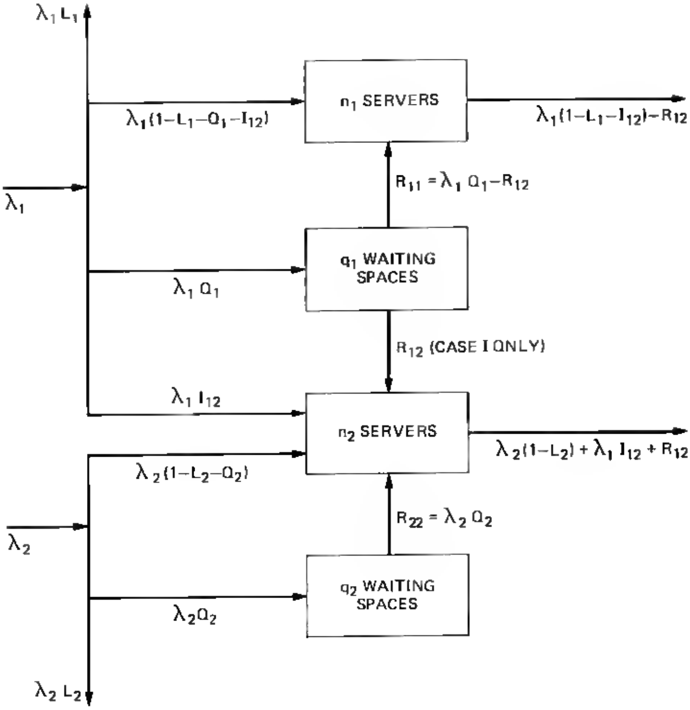

Fig. 1—Mean flow rates for an overflow system with queuing.

If all n_1 servers in the primary are busy when a demand from S_1 arrives and there is a free server in the secondary and no demands waiting in the secondary queue, the demand is served in the secondary. If there are no free servers, then the demand is queued in the primary queue, if one of the q_1 waiting spaces is available; otherwise, it is lost. Two different cases were considered for the treatment of demands that are waiting in the primary queue. In case I, a demand waiting in the primary queue may enter service either in the primary, when a server becomes free, or in the secondary, if a server becomes free and no demands are waiting in the secondary queue. In case II, no overflow is permitted from the primary queue, so that a demand in the primary queue must wait for a server in the primary to become free.

In this paper, we use an alternate approach to analyze the same overflow system. This alternate approach is useful, in particular, when q_2 is large, or even infinite. Which approach is preferable depends, essentially, on tbe relative size of q_2 to n_1. Other approaches to the problem were discussed in the earlier paper.[1] Let p_{ij} denote the steady-state probability that there are i demands in the primary and j demands in the secondary, either in service or waiting. These proba-

bilities satisfy a set of generalized birth-and-death equations, which take the form of partial difference equations connecting nearest neighboring states. Here we carry out an analysis that reduces the dimensions of the problem, which may be considerable in cases of interest. An analogous reduction was obtained in the earlier analysis.[1]

The basic technique is to separate variables in regions away from certain boundaries of the state space, the elements of which are (i, j). These regions are depicted in Fig. 2a. The analogous regions corresponding to the analysis of the earlier paper[1] are depicted in Fig. 2b, for comparison. The separation of variables leads to two sets of eigenvalue problems for the separation constant. The eigenvalues are roots of polynomial equations. The probabilities p_{ij} are then repre-

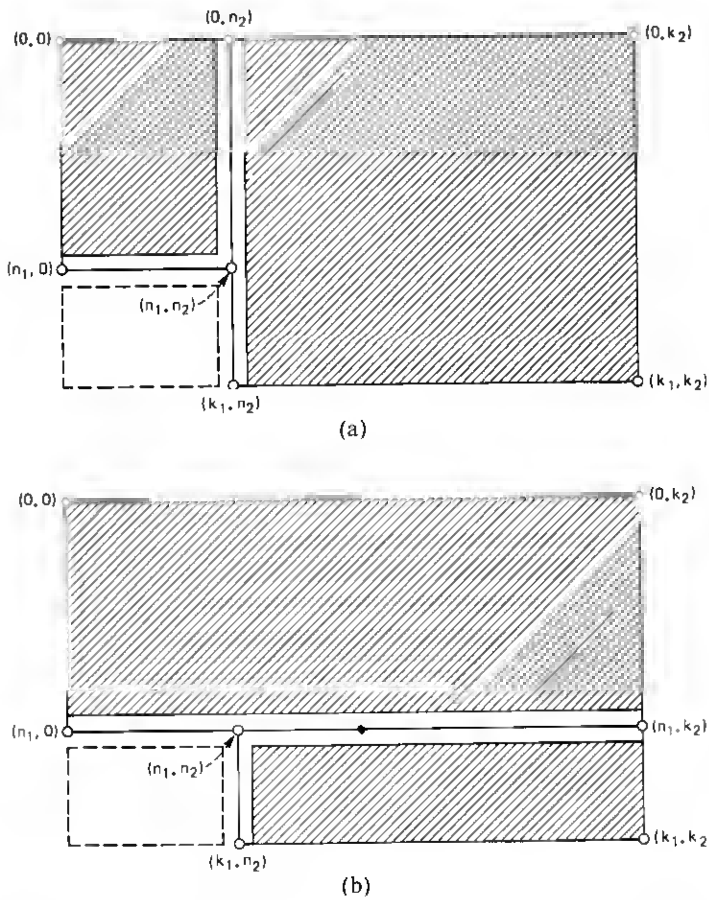

Fig. 2—Boundaries of regions in state space for the analysis of (a) this paper, and (b) the earlier paper.

sented in terms of the corresponding eigenfunctions. The constant coefficients in these representations are determined from the boundary conditions and the normalization condition that the sum of the probabilities is unity. In general, these constants have to be determined numerically. In case II, additional accessible states are in the region bordered by the broken lines in Fig. 2, since in this case demands wait in the primary queue even if a server is free in the secondary. In this additional region, the probabilities p_{ij} are expressed, as in the earlier analysis,[1] in terms of a fundamental solution of a partial difference equation.

There are various steady-state quantities of interest, which may be expressed in terms of the probabilities p_{ij}. The quantities include the loss (or blocking) probabilities, the average waiting times in the queues, and the average number of demands in service in each group. These quantities may be expressed directly in terms of the constant coefficients which occur in the representations for the probabilities p_{ij}. Thus the steady-state quantities of interest may be calculated directly, once the coefficients have been determined from the boundary and normalization conditions, without having to calculate the probabilities p_{ij}. Here again the reduction in the dimensions of the problem is valuable.

We first consider case I. The representation of the probabilities p_{ij} in terms of the eigenfunctions is discussed in Section II, and the boundary and normalization conditions are considered in Section III. Various steady-state quantities of interest are calculated in Section IV. Next, we consider case II and discuss the representation of the probabilities p_{ij} and the boundary and normalization conditions in Section V. The corresponding steady-state quantities of interest are considered in Section VI. Properties of the eigenfunctions which occur in the representations of the probabilities p_{ij} are given in the appendix.

II. REPRESENTATION OF SOLUTION: CASE I

Let p_{ij} denote the steady-state probability that there are i demands in the primary and j demands in the secondary, either in service or waiting. These probabilities satisfy a set of generalized birth-and-death equations,[2] which may be derived in a straightforward manner. We define the traffic intensities

$$a_1 = \lambda_1/\mu, \qquad a_2 = \lambda_2/\mu, \tag{1}$$

and let

$$k_1 = n_1 + q_1, \qquad k_2 = n_2 + q_2. \tag{2}$$

It is convenient to introduce the function

$$\chi_\ell = \begin{cases} 1, & \ell \geq 0, \\ 0, & \ell < 0, \end{cases} \tag{3}$$

as well as the Kronecker delta

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{4}$$

Then,[1] for case I, it is found that

$$\begin{aligned}
&[a_1(1-\delta_{ik_1}\chi_{j-n_2}) + a_2(1-\delta_{jk_2}) + \min(i, n_1) + \min(j, n_2)]p_{ij} \\
&\quad = (1-\chi_{i-1-n_1}\chi_{n_2-1-j})[a_1(1-\delta_{i0})p_{i-1,j} \\
&\qquad\qquad + (1-\delta_{jk_2})\min(j+1, n_2)p_{i,j+1}] \\
&\qquad + (1-\delta_{j0})[a_1\delta_{in_1}\chi_{n_2-j} + a_2(1-\chi_{i-1-n_1}\chi_{n_2-j})]p_{i,\,j-1,} \\
&\qquad + (1-\delta_{ik_1})[(1-\chi_{i-n_1}\chi_{n_2-1-j})\min(i+1, n_1) + n_2\chi_{i-n_1}\delta_{jn_2}]p_{i+1,j,}
\end{aligned} \tag{5}$$

for $0 \leq i \leq k_1$, $0 \leq j \leq k_2$. These equations were constructed to imply that

$$p_{ij} = 0, \qquad n_1 + 1 \leq i \leq k_1, \qquad 0 \leq j \leq n_2 - 1, \tag{6}$$

since it is impossible for demands to be waiting in the primary queue when a server is free in the secondary. The normalization condition is

$$\sum_{i=0}^{n_1}\sum_{j=0}^{n_2-1} p_{ij} + \sum_{i=0}^{k_1}\sum_{j=n_2}^{k_2} p_{ij} = 1. \tag{7}$$

We assume that $q_2 \geq 1$. Then, for $0 \leq i \leq k_1$ and $n_2 + 1 \leq j \leq k_2$, the variables in (5) may be separated, and there are solutions of the form $\alpha_i\beta_j$ where

$$\begin{aligned}
&[a_1(1-\delta_{ik_1}) + \min(i, n_1) + \kappa]\alpha_i \\
&\qquad = a_1(1-\delta_{i0})\alpha_{i-1} + (1-\delta_{ik_1})\min(i+1, n_1)\alpha_{i+1}
\end{aligned} \tag{8}$$

for $0 \leq i \leq k_1$, and

$$[a_2(1-\delta_{jk_2}) + n_2 - \kappa]\beta_j = a_2\beta_{j-1} + n_2(1-\delta_{jk_2})\beta_{j+1} \tag{9}$$

for $n_2 + 1 \leq j \leq k_2$, and κ is a separation constant. The solution of (9) may be expressed in terms of Chebyshev polynomials of the second kind,[3] $U_\ell(x)$. It is convenient to define

$$\Psi_\ell(\kappa) = \left(\frac{n_2}{a_2}\right)^{\ell/2} U_\ell\left(\frac{a_2 + n_2 - \kappa}{2\sqrt{a_2 n_2}}\right), \tag{10}$$

and

$$\phi_j(\kappa) = \Psi_{k_2-j}(\kappa) - \Psi_{k_2-j-1}(\kappa). \tag{11}$$

The properties of these functions which we will need are given in the appendix. We note here, however, that $U_0(x) \equiv 1$, $U_{-1}(x) \equiv 0$ and

$\phi_{k_2}(\kappa) \equiv 1$. It follows from (9) and (102) that β_j is proportional to $\phi_j(\kappa)$ for $n_2 \le j \le k_2$.

For $0 \le i \le n_1 - 1$, (8) implies that

$$(a_1 + i + \kappa)\alpha_i = a_1(1 - \delta_{i0})\alpha_{i-1} + (i + 1)\alpha_{i+1}. \tag{12}$$

The solution of (12) may be expressed in terms of Poisson-Charlier polynomials.[4,5] We here denote the solution of (12) for which $\alpha_0 = 1$ by $s_i(\kappa, a_1)$. The properties of $s_i(\kappa, a)$ which we will need are given in the appendix. For $n_1 \le i \le k_1$, (8) implies that

$$[a_1(1 - \delta_{ik_1}) + n_1 + \kappa]\alpha_i = a_i\alpha_{i-1} + n_1(1 - \delta_{ik_1})\alpha_{i+1}. \tag{13}$$

Corresponding to (10) and (11), we define

$$\Omega_\ell(\kappa) = \left(\frac{n_1}{a_1}\right)^{\ell/2} U_\ell\left(\frac{a_1 + n_1 + \kappa}{2\sqrt{a_1 n_1}}\right) \tag{14}$$

and

$$\theta_i(\kappa) = \Omega_{k_1-i}(\kappa) - \Omega_{k_1-i-1}(\kappa). \tag{15}$$

It follows from (13) and (111) that α_i is proportional to $\theta_i(\kappa)$ for $n_1 - 1 \le i \le k_1$, and we note that $\theta_{k_1}(\kappa) \equiv 1$.

Consequently, we take

$$\alpha_i = \begin{cases} s_i(\kappa, a_1)\theta_{n_1}(\kappa), & 0 \le i \le n_1, \\ s_{n_1}(\kappa, a_1)\theta_i(\kappa), & n_1 - 1 \le i \le k_1, \end{cases} \tag{16}$$

where

$$s_{n_1-1}(\kappa, a_1)\theta_{n_1}(\kappa) = s_{n_1}(\kappa, a_1)\theta_{n_1-1}(\kappa). \tag{17}$$

With the help of (15), (96), (97), and (110), this equation may be written in the form

$$\kappa[s_{n_1}(1 + \kappa, a_1)\Omega_{q_1}(\kappa) - s_{n_1-1}(1 + \kappa, a_1)\Omega_{q_1-1}(\kappa)] = 0. \tag{18}$$

The expression in the square brackets in (18) is a polynomial in κ of degree $k_1 = n_1 + q_1$. It was shown[1] that its zeros are negative and distinct, and we denote them by κ_r, $r = 1, \cdots, k_1$. We also let $\kappa_0 = 0$.

It was also shown[1] that the zeros of $\phi_{n_2}(\kappa)$ are positive. Hence, $\phi_{n_2}(\kappa_r) \ne 0$, $r = 0, \cdots, k_1$. Moreover, it follows from (118) that

$$\lim_{q_2 \to \infty} \frac{\phi_j(\kappa_r)}{\phi_{n_2}(\kappa_r)} = \left[\sqrt{\frac{a_2}{n_2}} \frac{1}{(\epsilon_r + \sqrt{\epsilon_r^2 - 1})}\right]^{j-n_2}, \qquad r = 1, \cdots, k_1, \tag{19}$$

for fixed j, where

$$\epsilon_r = \frac{(a_2 + n_2 - \kappa_r)}{2\sqrt{a_2 n_2}}, \tag{20}$$

and the positive square root of $\epsilon_r^2 - 1$ is taken in (19). We note, from (109), that

$$\epsilon_r + \sqrt{\epsilon_r^2 - 1} > \sqrt{\frac{a_2}{n_2}}, \qquad r = 1, \cdots, k_1, \tag{21}$$

since $\kappa_r < 0$, $r = 1, \cdots, k_1$. On the other hand, from (105),

$$\frac{\phi_j(0)}{\phi_{n_2}(0)} = \left(\frac{a_2}{n_2}\right)^{j-n_2}, \tag{22}$$

and, as expected, the condition $a_2 < n_2$ is necessary for a steady-state solution when $q_2 = \infty$.

In view of the above results, we may represent the probabilities p_{ij}, for $n_2 \le j \le k_2$, in the form

$$p_{ij} = \begin{cases} \sum_{r=0}^{k_1} b_r s_i(\kappa_r, a_1) s_{n_2}(-\kappa_r, a_2) \theta_{n_1}(\kappa_r) \dfrac{\phi_j(\kappa_r)}{\phi_{n_2}(\kappa_r)}, & 0 \le i \le n_1, \\ \sum_{r=0}^{k_1} b_r s_{n_1}(\kappa_r, a_1) s_{n_2}(-\kappa_r, a_2) \theta_i(\kappa_r) \dfrac{\phi_j(\kappa_r)}{\phi_{n_2}(\kappa_r)}, & n_1 \le i \le k_1, \end{cases} \tag{23}$$

where the constants b_r are to be determined. The reason for the factor $s_{n_2}(-\kappa_r, a_2)$ will be apparent shortly. We note, from (94), that $s_{n_2}(-\kappa_r, a_2) > 0$, since $\kappa_r \le 0$, $r = 0, \cdots, k_1$.

For $0 \le i \le n_1 - 1$ and $0 \le j \le n_2 - 1$, the variables in (5) may be separated, and there are solutions of the form $\gamma_i \delta_j$ where

$$(a_1 + i + \eta)\gamma_i = a_1(1 - \delta_{i0})\gamma_{i-1} + (i+1)\gamma_{i+1}, \qquad 0 \le i \le n_1 - 1, \tag{24}$$

and

$$(a_2 + j - \eta)\delta_j = a_2(1 - \delta_{j0})\delta_{j-1} + (j+1)\delta_{j+1}, \qquad 0 \le j \le n_2 - 1, \tag{25}$$

and η is a separation constant. It follows from (93) that γ_i is proportional to $s_i(\eta, a_1)$ for $0 \le i \le n_1$ and δ_j is proportional to $s_j(-\eta, a_2)$ for $0 \le j \le n_2$. Now $s_{n_2}(-\eta, a_2)$ is a polynomial in η of degree n_2. It was shown[1] that its zeros are positive and distinct, and we denote them by η_ℓ, $\ell = 1, \cdots, n_2$. Then

$$s_{n_2}(-\eta_\ell, a_2) = 0, \qquad \ell = 1, \cdots, n_2. \tag{26}$$

Consequently, we represent p_{ij}, for $0 \le i \le n_1$ and $0 \le j \le n_2$, in the form

$$p_{ij} = \sum_{r=0}^{k_1} b_r s_i(\kappa_r, a_1) s_j(-\kappa_r, a_2) \theta_{n_1}(\kappa_r) + \sum_{\ell=1}^{n_2} c_\ell s_i(\eta_\ell, a_1) s_j(-\eta_\ell, a_2), \tag{27}$$

where the constants c_ℓ are also to be determined. Note that the representations in (23) and (27) agree for $j = n_2$, $0 \le i \le n_1$, in view of (26).

It remains to satisfy the boundary conditions at $i = n_1$, $0 \le j \le n_2 - 1$ and at $j = n_2$, $0 \le i \le k_1$, as well as the normalization condition (7). This is done in the next section.

III. BOUNDARY CONDITIONS: CASE I

From (5), the boundary conditions at $i = n_1$ imply that

$$(a_1 + a_2 + n_1 + j)p_{n_1,j} = a_1 p_{n_1-1,j} + (a_1 + a_2)(1 - \delta_{j0})p_{n_1,j-1} + (j+1)p_{n_1,j+1}, \quad (28)$$

for $0 \le j \le n_2 - 1$. Also, the boundary conditions at $j = n_2$ imply that

$$(a_1 + a_2 + i + n_2)p_{i,n_2}, = a_1(1 - \delta_{i0})p_{i-1,n_2} + a_2 p_{i,n_2-1} + (i+1)p_{i+1,n_2} + n_2 p_{i,n_2+1}, \quad (29)$$

for $0 \le i \le n_1 - 1$,

$$[a_1(1 - \delta_{q_1,0}) + a_2 + n_1 + n_2]p_{n_1,n_2} = a_1 p_{n_1-1,n_2} + (a_1 + a_2)p_{n_1,n_2-1} + (n_1 + n_2)(1 - \delta_{q_1,0})p_{n_1+1,n_2} + n_2 p_{n_1,n_2+1}, \quad (30)$$

and, if $q_1 \ge 1$,

$$[a_1(1 - \delta_{ik_1}) + a_2 + n_1 + n_2]p_{i,n_2} = a_1 p_{i-1,n_2} + (n_1 + n_2)(1 - \delta_{ik_1})p_{i+1,n_2} + n_2 p_{i,n_2+1}, \quad (31)$$

for $n_1 + 1 \le i \le k_1$.

If we substitute (27) into (28), we find, after reduction with the help of the recurrence relations in the appendix, that

$$\sum_{r=0}^{k_1} b_r \theta_{n_1}(\kappa_r)[\kappa_r s_{n_1}(1 + \kappa_r, a_1)s_j(-\kappa_r, a_2) + a_1 s_{n_1}(\kappa_r, a_1)s_j(-1 - \kappa_r, a_2)] + \sum_{\ell=1}^{n_2} c_\ell[\eta_\ell s_{n_1}(1 + \eta_\ell, a_1)s_j(-\eta_\ell, a_2) + a_1 s_{n_1}(\eta_\ell, a_1)s_j(-1 - \eta_\ell, a_2)] = 0, \quad (32)$$

for $0 \le j \le n_2 - 1$. We remark that the first sum in (32) may be written in a different form with the help of the relationship

$$\kappa_r[s_{n_1}(1 + \kappa_r, a_1)\theta_{n_1}(\kappa_r) + s_{n_1}(\kappa_r, a_1)\Omega_{q_1-1}(\kappa_r)] = 0, \quad (33)$$

which follows from (15), (97), and the fact that κ_r, $r = 0, \cdots, k_1$, are the roots of (18).

If we make use of (23) and (27) in (29), we find after reduction that

$$a_2 \sum_{r=0}^{k_1} b_r s_i(\kappa_r, a_1)\theta_{n_1}(\kappa_r)\left[s_{n_2}(-\kappa_r, a_2)\frac{\phi_{n_2-1}(\kappa_r)}{\phi_{n_2}(\kappa_r)} - s_{n_2-1}(-\kappa_r, a_2)\right] - a_2 \sum_{\ell=1}^{n_2} c_\ell s_i(\eta_\ell, a_1)s_{n_2-1}(-\eta_\ell, a_2) = 0 \quad (34)$$

for $0 \le i \le n_1 - 1$. The first sum in (34) may be written in a different form with the help of the relationship

$$a_2[s_{n_2}(-\kappa_r, a_2)\phi_{n_2-1}(\kappa_r) - s_{n_2-1}(-\kappa_r, a_2)\phi_{n_2}(\kappa_r)]$$
$$= -\kappa_r[s_{n_2}(1-\kappa_r, a_2)\Psi_{q_2}(\kappa_r) - s_{n_2-1}(1-\kappa_r, a_2)\Psi_{q_2-1}(\kappa_r)], \qquad (35)$$

which follows from (11) and the recurrence relations in the appendix. We note that $\kappa_0 = 0$. Also, for $q_1 \ge 1$, if we make use of (23) in (31), it is found after reduction that

$$\sum_{r=0}^{k_1} b_r s_{n_1}(\kappa_r, a_1)s_{n_2}(-\kappa_r, a_2)\left[a_2\theta_i(\kappa_r)\frac{\phi_{n_2-1}(\kappa_r)}{\phi_{n_2}(\kappa_r)} - n_2(1-\delta_{ik_1})\theta_{i+1}(\kappa_r)\right] = 0, \qquad (36)$$

for $n_1 + 1 \le i \le k_1$.

If we substitute (23) and (27) into (30), and make use of (17), (102), and (111), we find that

$$\sum_{r=0}^{k_1} b_r s_{n_1}(\kappa_r, a_1)s_{n_2}(-\kappa_r, a_2)\left[a_2\theta_{n_1}(\kappa_r)\frac{\phi_{n_2-1}(\kappa_r)}{\phi_{n_2}(\kappa_r)} - n_2(1-\delta_{q_1,0})\theta_{n_1+1}(\kappa_r)\right]$$
$$- (a_1 + a_2)\sum_{r=0}^{k_1} b_r s_{n_1}(\kappa_r, a_1)s_{n_2-1}(-\kappa_r, a_2)\theta_{n_1}(\kappa_r)$$
$$- (a_1 + a_2)\sum_{\ell=1}^{n_2} c_\ell s_{n_1}(\eta_\ell, a_1)s_{n_2-1}(-\eta_\ell, a_2) = 0. \qquad (37)$$

This condition may be shown to be redundant, by summing (32) from $j = 0$ to $n_2 - 1$, (34) from $i = 0$ to $n_1 - 1$ and, for $q_1 \ge 1$, (36) from $i = n_1 + 1$ to k_1, and adding.

The constants b_r, $r = 0, \cdots k_1$ and c_ℓ, $\ell = 1, \cdots, n_2$ are determined by (32), (34), and (36) only to within a multiplicative constant, which is determined from the normalization condition (7). But, from (23), with the help of (15), (18), (97), and (98), it is found that

$$\sum_{i=0}^{k_1} p_{ij} = b_0 s_{n_2}(0, a_2)\frac{\phi_j(0)}{\phi_{n_2}(0)}[s_{n_1}(1, a_1)\Omega_{q_1}(0) - s_{n_1-1}(1, a_1)\Omega_{q_1-1}(0)], \qquad (38)$$

for $n_2 \le j \le k_2$. Then, from (7) and (27), with the help of (11) and (98), it follows that

$$b_0 s_{n_2}(0, a_2)\frac{\Psi_{q_2}(0)}{\phi_{n_2}(0)}[s_{n_1}(1, a_1)\Omega_{q_1}(0) - s_{n_1-1}(1, a_1)\Omega_{q_1-1}(0)]$$
$$+ \sum_{r=0}^{k_1} b_r s_{n_1}(1+\kappa_r, a_1)s_{n_2-1}(1-\kappa_r, a_2)\theta_{n_1}(\kappa_r)$$
$$+ \sum_{\ell=1}^{n_2} c_\ell s_{n_1}(1+\eta_\ell, a_1)s_{n_2-1}(1-\eta_\ell, a_2) = 1. \qquad (39)$$

We note, from (104) and (105), that

$$\lim_{q_2\to\infty} \frac{\Psi_{q_2}(0)}{\phi_{n_2}(0)} = \frac{n_2}{(n_2 - a_2)}, \quad \text{for} \quad a_2 < n_2. \tag{40}$$

Once the constants b_r and c_r have been determined, the steady-state probabilities p_{ij} may be calculated from (23) and (27). We remark that the number of constants to be determined is only $k_1 + 1 + n_2$, whereas the number of probabilities p_{ij} is $(q_2 + 1)(k_1 + 1) + n_2(n_1 + 1)$, which in general is considerably larger. It is of interest to compare the approach taken in this paper with that adopted in the earlier paper[1] (see Fig. 2). There the variables in (5) were separated in the region $0 \le i \le n_1 - 1$ and $0 \le j \le k_2$, leading to $k_2 + 1$ eigenvalues and eigenfunctions, and, for $q_1 \ge 1$, in the region $n_1 + 1 \le i \le k_1$ and $n_2 + 1 \le j \le k_2$, leading to q_1 eigenvalues with corresponding eigenfunctions vanishing at $i = n_1$. In the corresponding representation of the probabilities p_{ij} in terms of the eigenfunctions, there are thus $k_2 + 1 + q_1$ constants to be determined. In view of (2), which approach is preferable depends on the relative size of q_2 to n_1. The approach adopted in this paper is suitable if q_2 is large, or even infinite.

IV. SOME STEADY-STATE QUANTITIES: CASE I

We proceed now to the calculation of various steady-state quantities of interest. These quantities are depicted in Fig. 1, which indicates the mean flow rates. The loss probabilities L_1 and L_2 are given by

$$L_1 = \sum_{j=n_2}^{k_2} p_{k_1,j}, \quad L_2 = \sum_{i=0}^{k_1} p_{i}, k_2, \tag{41}$$

and the probabilities that a demand from the primary, or secondary, source is queued on arrival are

$$Q_1 = (1 - \delta_{q_1,0}) \sum_{i=n_1}^{k_1-1} \sum_{j=n_2}^{k_2} p_{ij}, \quad Q_2 = \sum_{i=0}^{k_1} \sum_{j=n_2}^{k_2-1} p_{ij}. \tag{42}$$

The probability that a demand arriving from the primary source overflows immediately is

$$I_{12} = \sum_{j=0}^{n_2-1} p_{n_1,j}. \tag{43}$$

Since the mean service rate is μ, the mean departure rate from the primary queue to the primary servers is

$$R_{11} = n_1\mu(1 - \delta_{q_1,0}) \sum_{i=n_1+1}^{k_1} \sum_{j=n_2}^{k_2} p_{ij}, \tag{44}$$

while the mean rate of overflow from the primary queue to the

secondary servers is

$$R_{12} = n_2\mu(1 - \delta_{q_1,0}) \sum_{i=n_1+1}^{k_1} p_{i,n_2}. \tag{45}$$

The mean departure rate from the secondary queue is

$$R_{22} = n_2\mu \sum_{i=0}^{k_1} \sum_{j=n_2+1}^{k_2} p_{ij}. \tag{46}$$

It may be verified from (5) that

$$R_{11} + R_{12} = \lambda_1 Q_1, \quad R_{22} = \lambda_2 Q_2. \tag{47}$$

These relationships hold since, in the steady state, the departure rates from the queues are equal to the arrival rates to them.

The average number of demands in the primary and secondary queues are

$$V_1 = \sum_{i=n_1}^{k_1} \sum_{j=n_2}^{k_2} (i - n_1)p_{ij}, \quad V_2 = \sum_{i=0}^{k_1} \sum_{j=n_2+1}^{k_2} (j - n_2)p_{ij}. \tag{48}$$

Also, the average number of demands in service in the two groups are

$$X_1 = \sum_{i=0}^{n_1} \sum_{j=0}^{k_2} ip_{ij} + n_1(1 - \delta_{q_1,0}) \sum_{i=n_1+1}^{k_1} \sum_{j=n_2}^{k_2} p_{ij} \tag{49}$$

and

$$X_2 = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2-1} jp_{ij} + n_2 \sum_{i=0}^{k_1} \sum_{j=n_2}^{k_2} p_{ij}. \tag{50}$$

If we apply Little's theorem[2] to the primary and secondary queues, we find that the average waiting times of the demands which are queued in the primary or in the secondary are given by

$$W_1 = \frac{V_1}{\lambda_1 Q_1}\,(q_1 \geq 1), \quad W_2 = \frac{V_2}{\lambda_2 Q_2}, \tag{51}$$

respectively, independently of the order of service within each queue. Also, if we apply Little's theorem to the primary and secondary groups of servers, we obtain

$$\lambda_1(1 - L_1 - I_{12}) - R_{12} = \mu X_1, \quad \lambda_2(1 - L_2) + \lambda_1 I_{12} + R_{12} = \mu X_2, \tag{52}$$

since the mean service rate is μ.

The steady-state quantities of interest may be expressed in terms of the constants b_r and c_ℓ with the help of the representations in (23) and (27). From (41), it is found, with the help of (11), since $\theta_{k_1}(\kappa) \equiv 1$, that

$$L_1 = \sum_{r=0}^{k_1} b_r s_{n_1}(\kappa_r, a_1) s_{n_2}(-\kappa_r, a_2) \frac{\Psi_{q_2}(\kappa_r)}{\phi_{n_2}(\kappa_r)}, \tag{53}$$

and, with the help of (22) and (38), that

$$L_2 = b_0\left(\frac{a_2}{n_2}\right)^{q_2} s_{n_2}(0, a_2)[s_{n_1}(1, a_1)\Omega_{q_1}(0) - s_{n_1-1}(1, a_1)\Omega_{q_1-1}(0)]. \quad (54)$$

From (42), it is found, with the help of (11) and (15), that

$$Q_1 = \sum_{r=0}^{k_1} b_r s_{n_1}(\kappa_r, a_1) s_{n_2}(-\kappa_r, a_2)[\Omega_{q_1}(\kappa_r) - 1]\frac{\Psi_{q_2}(\kappa_r)}{\phi_{n_2}(\kappa_r)}, \quad (55)$$

and, with the help of (38), that

$$Q_2 = b_0 \frac{[\Psi_{q_2}(0) - 1]}{\phi_{n_2}(0)} \cdot s_{n_2}(0, a_2)[s_{n_1}(1, a_1)\Omega_{q_1}(0) - s_{n_1-1}(1, a_1)\Omega_{q_1-1}(0)]. \quad (56)$$

Next, from (43), with the help of (98), it is found that

$$I_{12} = \sum_{r=0}^{k_1} b_r s_{n_1}(\kappa_r, a_1) s_{n_2-1}(1 - \kappa_r, a_2)\theta_{n_1}(\kappa_r) + \sum_{\ell=1}^{n_2} c_\ell s_{n_1}(\eta_\ell, a_1) s_{n_2-1}(1 - \eta_\ell, a_2). \quad (57)$$

Also, from (44) and (45), if we make use of (11) and (15), we obtain

$$R_{11} = n_1\mu \sum_{r=0}^{k_1} b_r s_{n_1}(\kappa_r, a_1) s_{n_2}(-\kappa_r, a_2)\Omega_{q_1-1}(\kappa_r)\frac{\Psi_{q_2}(\kappa_r)}{\phi_{n_2}(\kappa_r)} \quad (58)$$

and

$$R_{12} = n_2\mu \sum_{r=0}^{k_1} b_r s_{n_1}(\kappa_r, a_1) s_{n_2}(-\kappa_r, a_2)\Omega_{q_1-1}(\kappa_r). \quad (59)$$

From (46), with the help of (11), (38), (56), and (104), it follows that $R_{22} = \lambda_2 Q_2$, as expected. The first relationship in (47) provides a useful numerical check, in view of the expressions in (55), (58), and (59).

From (48), with the help of (11), (114), and (115), it is found that

$$V_1 = b_0 s_{n_1}(0, a_1) s_{n_2}(0, a_2)\left(\frac{n_1}{a_1}\right)^{q_1} \Lambda_{q_1}\left(\frac{a_1}{n_1}\right)\frac{\Psi_{q_2}(0)}{\phi_{n_2}(0)} + \sum_{r=1}^{k_1} \frac{b_r}{\kappa_r} s_{n_1}(\kappa_r, a_1) s_{n_2}(-\kappa_r, a_2) \cdot \{a_1[\Omega_{q_1}(\kappa_r) - 1] - n_1\Omega_{q_1-1}(\kappa_r)\}\frac{\Psi_{q_2}(\kappa_r)}{\phi_{n_2}(\kappa_r)}, \quad (60)$$

where

$$\Lambda_q(\xi) = \sum_{\ell=1}^{q} \ell\xi^\ell = \begin{cases} \xi[1 - (q+1)\xi^q + q\xi^{q+1}]/(1-\xi)^2, & \xi \neq 1 \\ \tfrac{1}{2}q(q+1), & \xi = 1. \end{cases} \quad (61)$$

Also, with the help of (22) and (38), it is found that

$$V_2 = b_0 \Lambda_{q_2}\left(\frac{a_2}{n_2}\right) s_{n_2}(0, a_2)[s_{n_1}(1, a_1)\Omega_{q_1}(0) - s_{n_1-1}(1, a_1)\Omega_{q_1-1}(0)]. \quad (62)$$

It follows from (51), (56), and (62) that

$$W_2 = \frac{\Lambda_{q_2}(a_2/n_2)\phi_{n_2}(0)}{\lambda_2[\Psi_{q_2}(0) - 1]}. \quad (63)$$

Next, from (49), with the help of (7), (11), (98), and (99), it is found that

$$X_1 = n_1 - \sum_{\ell=1}^{n_2} c_\ell s_{n_1-1}(2 + \eta_\ell, a_1) s_{n_2-1}(1 - \eta_\ell, a_2) - \sum_{r=0}^{k_1} b_r s_{n_1-1}(2 + \kappa_r, a_1)\theta_{n_1}(\kappa_r) \cdot \left[s_{n_2-1}(1 - \kappa_r, a_2) + s_{n_2}(-\kappa_r, a_2) \frac{\Psi_{q_2}(\kappa_r)}{\phi_{n_2}(\kappa_r)} \right]. \quad (64)$$

Similarly, from (50), it is found that

$$X_2 = n_2 - \sum_{\ell=1}^{n_2} c_\ell s_{n_1}(1 + \eta_\ell, a_1) s_{n_2-1}(2 - \eta_\ell, a_2) - \sum_{r=0}^{k_1} b_r s_{n_1}(1 + \kappa_r, a_1) s_{n_2-1}(2 - \kappa_r, a_2)\theta_{n_1}(\kappa_r). \quad (65)$$

In view of (53), (54), (57), (59), (64), and (65), the relationships in (52) provide a useful numerical check.

From (107) to (109), it follows that

$$\lim_{q_2 \to \infty} \frac{\Psi_{q_2}(\kappa_r)}{\phi_{n_2}(\kappa_r)} = \frac{(\epsilon_r + \sqrt{\epsilon_r^2 - 1})}{(\epsilon_r + \sqrt{\epsilon_r^2 - 1} - \sqrt{a_2/n_2})}, \quad r = 1, \cdots, k_1, \quad (66)$$

where ϵ_r is given by (20). With the help of (40) and (66), we obtain the results for the limiting case $q_2 = \infty$, with $a_2 < n_2$, in (53), (55), (58), (60), and (64). From (54), as expected, $L_2 \to 0$ as $q_2 \to \infty$, for $a_2 < n_2$. Also, since $\phi_{k_2}(0) = 1$, it follows from (22) and (40) that

$$\lim_{q_2 \to \infty} \frac{[\Psi_{q_2}(0) - 1]}{\phi_{n_2}(0)} = \frac{n_2}{(n_2 - a_2)}, \quad \text{for } a_2 < n_2, \quad (67)$$

and from (61) that

$$\lim_{q_2 \to \infty} \Lambda_{q_2}\left(\frac{a_2}{n_2}\right) = \frac{n_2 a_2}{(n_2 - a_2)^2}, \quad \text{for } a_2 < n_2. \quad (68)$$

From (67) and (68), we obtain the limiting results in (56) and (62), and

from (1) and (63), we have

$$\lim_{q_2 \to \infty} W_2 = \frac{1}{\mu(n_2 - a_2)}, \quad \text{for } a_2 < n_2. \tag{69}$$

We remark that the steady-state quantities of interest may be calculated directly, once the constants b_r and c_ℓ have been determined, without having to calculate the probabilities p_{ij}.

V. REPRESENTATION AND BOUNDARY CONDITIONS: CASE II

We now consider the second case, in which no overflow is permitted from the primary queue, so that a demand in the primary queue must wait for a server in the primary to become free. Since case II differs from case I only in the treatment of demands waiting in the primary queue, we assume that $q_1 \geq 1$, as well as $q_2 \geq 1$. The steady-state probabilities p_{ij} now satisfy the equations[1]

$$\begin{aligned}
&[a_1(1 - \delta_{ik_1}\chi_{j-n_2}) + a_2(1 - \delta_{jk_2}) + \min(i, n_1) + \min(j, n_2)]p_{ij} \\
&\qquad = a_1(1 - \delta_{i0})(1 - \chi_{i-n_1-1}\chi_{n_2-1-j})p_{i-1,j} \\
&\qquad\quad + (1 - \delta_{j0})(a_1\chi_{i-n_1}\chi_{n_2-j} + a_2)p_{i,j-1} \\
&\qquad\quad + (1 - \delta_{ik_1})\min(i + 1, n_1)p_{i+1,j} \\
&\qquad\quad + (1 - \delta_{jk_2})\min(j + 1, n_2)p_{i,j+1},
\end{aligned} \tag{70}$$

for $0 \leq i \leq k_1$, $0 \leq j \leq k_2$. The normalization condition is

$$\sum_{i=0}^{k_1} \sum_{j=0}^{k_2} p_{ij} = 1. \tag{71}$$

Now (5) and (70) are identical for $0 \leq i \leq n_1 - 1$, $0 \leq j \leq n_2$ and for $0 \leq i \leq k_1$, $n_2 + 1 \leq j \leq k_2$. It follows that the representations in (23) and (27) are still valid, although the constants b_r and c_ℓ will, of course, be different. However, the boundary conditions (29), which correspond to $j = n_2$ for $0 \leq i \leq n_1 - 1$, are still satisfied, and hence (34) still holds, for $0 \leq i \leq n_1 - 1$. The remaining boundary conditions are different. Also, from (70), for $n_1 + 1 \leq i \leq k_1$ and $0 \leq j \leq n_2 - 1$, we have

$$\begin{aligned}
&(a_1 + a_2 + n_1 + j)p_{ij} \\
&\qquad = (a_1 + a_2)(1 - \delta_{j0})p_{i,j-1} + n_1(1 - \delta_{ik_1})p_{i+1,j} + (j + 1)p_{i,j+1}.
\end{aligned} \tag{72}$$

We define[1] the quantities Π_{mj}, for $m, j = 0, 1, \cdots$, as the solutions of the equations

$$\begin{aligned}
(a_1 + a_2 + n_1 + j)\Pi_{mj} &= (a_1 + a_2)(1 - \delta_{j0})\Pi_{m,j-1} \\
&\quad + n_1(1 - \delta_{m0})\Pi_{m-1,j} + (j + 1)\Pi_{m,j+1},
\end{aligned} \tag{73}$$

which satisfy the initial conditions

$$\Pi_{m0} = \delta_{m0}, \quad m = 0, 1, \cdots . \tag{74}$$

It follows from (73) and (74), by induction on j, that

$$\Pi_{mj} = 0, \quad m > j. \tag{75}$$

It may be verified, from (72) to (74), that

$$p_{ij} = \sum_{m=i}^{k_1} p_{m0}\Pi_{m-i,j}, \quad n_1 + 1 \le i \le k_1, \; 0 \le j \le n_2. \tag{76}$$

The quantities Π_{mj} may be calculated sequentially from the recurrence relations (73), with the help of (75) and the initial condition $\Pi_{00} = 1$.

In order that the representations in (23) and (76) agree for $j = n_2$, we must have

$$\sum_{m=i}^{k_1} p_{m0}\Pi_{m-i,n_2} = \sum_{r=0}^{k_1} b_r s_{n_1}(\kappa_r, a_1) s_{n_2}(-\kappa_r, a_2)\theta_i(\kappa_r) \tag{77}$$

for $n_1 + 1 \le i \le k_1$. But, from (73), (74), and (93), it follows that $\Pi_{0,n_2} = s_{n_2}(n_1, a_1 + a_2) > 0$. Hence, (77) may be used to solve successively for $p_{k_1,0}, \cdots, p_{n_1+1,0}$ in terms of b_r, $r = 0, \cdots, k_1$. It remains to satisfy the boundary conditions at $i = n_1$, $0 \le j \le n_2 - 1$ and at $j = n_2$, $n_1 \le i \le k_1$, as well as (34) for $0 \le i \le n_1 - 1$, and the normalization condition (71). From (70), these boundary conditions are

$$(a_1 + a_2 + n_1 + j)p_{n_1,j} = a_1 p_{n_1-1,j} + (a_1 + a_2)(1 - \delta_{j0})p_{n_1,j-1} + n_1 p_{n_1+1,j} + (j + 1)p_{n_1,j+1}, \tag{78}$$

for $0 \le j \le n_2 - 1$ and

$$[a_1(1 - \delta_{ik_1}) + a_2 + n_1 + n_2]p_{i,n_2} = a_1 p_{i-1,n_2} + (a_1 + a_2)p_{i,n_2-1} + n_1(1 - \delta_{ik_1})p_{i+1,n_2} + n_2 p_{i,n_2+1}, \tag{79}$$

for $n_1 \le i \le k_1$.

From (27), (76), and (78), we find, after reduction, that

$$\sum_{r=0}^{k_1} b_r\theta_{n_1}(\kappa_r)[\kappa_r s_{n_1}(1 + \kappa_r, a_1)s_j(-\kappa_r, a_2) + a_1 s_{n_1}(\kappa_r, a_1)s_j(-1 - \kappa_r, a_2)] + \sum_{\ell=1}^{n_2} c_\ell[\eta_\ell s_{n_1}(1 + \eta_\ell, a_1)s_j(-\eta_\ell, a_2) + a_1 s_{n_1}(\eta_\ell, a_1)s_j(-1 - \eta_\ell, a_2)] - n_1 \sum_{m=n_1+1}^{k_1} p_{m0}\Pi_{m-n_1-1,j} = 0, \tag{80}$$

for $0 \le j \le n_2 - 1$. Also, from (23), (76), and (79), it is found that

$$a_2 \sum_{r=0}^{k_1} b_r s_{n_1}(\kappa_r, a_1) s_{n_2}(-\kappa_r, a_2) \theta_i(\kappa_r) \frac{\phi_{n_2-1}(\kappa_r)}{\phi_{n_2}(\kappa_r)} - (a_1 + a_2) \sum_{m=i}^{k_1} p_{m0} \Pi_{m-i,n_2-1} = 0, \quad (81)$$

for $n_1 + 1 \le i \le k_1$.

From (23) and (27), the boundary condition corresponding to $i = n_1$ in (79) reduces to

$$a_2 \sum_{r=0}^{k_1} b_r s_{n_1}(\kappa_r, a_1) s_{n_2}(-\kappa_r, a_2) \theta_{n_1}(\kappa_r) \frac{\phi_{n_2-1}(\kappa_r)}{\phi_{n_2}(\kappa_r)} - (a_1 + a_2) \sum_{r=0}^{k_1} b_r s_{n_1}(\kappa_r, a_1) s_{n_2-1}(-\kappa_r, a_2) \theta_{n_1}(\kappa_r) - (a_1 + a_2) \sum_{\ell=1}^{n_2} c_\ell s_{n_1}(\eta_\ell, a_1) s_{n_2-1}(-\eta_\ell, a_2) = 0. \quad (82)$$

This condition may be shown to be redundant, by summing (80) from $j = 0$ to $n_2 - 1$, (34) from $i = 0$ to $n_1 - 1$, and (81) from $i = n_1 + 1$ to k_1, and adding.

The constants b_r, $r = 0, \cdots, k_1$ and c_ℓ, $\ell = 1, \cdots, n_2$, and the probabilities p_{m0}, $m = n_1 + 1, \cdots, k_1$, are determined by (34), (77), (80), and (81) only to within a multiplicative constant, which is determined by the normalization condition (71). But, from (70), if we sum on j from 0 to k_2, and on i from ℓ to k_1, we obtain

$$n_1 \sum_{j=0}^{k_2} p_{\ell j} = a_1 \sum_{j=n_2}^{k_2} p_{\ell-1,j}, \quad n_1 + 1 \le \ell \le k_1. \quad (83)$$

Hence, from (23), (27), and (83), with the help of (11), (15), and (98), the normalization condition (71) implies that

$$\sum_{r=0}^{k_1} b_r s_{n_1}(1 + \kappa_r, a_1) \theta_{n_1}(\kappa_r) \left[s_{n_2}(-\kappa_r, a_2) \frac{\Psi_{q_2}(\kappa_r)}{\phi_{n_2}(\kappa_r)} + s_{n_2-1}(1 - \kappa_r, a_2) \right] + \sum_{\ell=1}^{n_2} c_\ell s_{n_1}(1 + \eta_\ell, a_1) s_{n_2-1}(1 - \eta_\ell, a_2) + \frac{a_1}{n_1} \sum_{r=0}^{k_1} b_r s_{n_1}(\kappa_r, a_1) s_{n_2}(-\kappa_r, a_2) [\Omega_{q_1}(\kappa_r) - 1] \frac{\Psi_{q_2}(\kappa_r)}{\phi_{n_2}(\kappa_r)} = 1. \quad (84)$$

VI. SOME STEADY-STATE QUANTITIES: CASE II

We now consider the calculation of various steady-state quantities of interest. Since no overflow is permitted from the primary queue, $R_{12} = 0$ in Fig. 1. The loss probabilities L_1 and L_2 are given by (41), and

the probabilities Q_1 and Q_2 that demands from the primary and secondary sources are queued on arrival are given by (42). Hence, from (23) and (27), it follows that (53) to (56) still hold. We emphasize, however, that the constants b_r, as well as c_ℓ, differ for the two cases. Similarly, R_{22} is given by (46) and V_2 is given by (48), so that $R_{22} = \lambda_2 Q_2$, as before, and (62) holds. Also,

$$X_1 = \sum_{i=0}^{k_1} \sum_{j=0}^{k_2} \min(i, n_1) p_{ij} = n_1 - \sum_{i=0}^{n_1} \sum_{j=0}^{k_2} (n_1 - i) p_{ij}, \tag{85}$$

and it follows, from (23) and (27), that (64) holds.

The probability that a demand arriving from the primary source overflows (immediately) is

$$I_{12} = \sum_{i=n_1}^{k_1} \sum_{j=0}^{n_2-1} p_{ij} = \sum_{j=0}^{n_2-1} \left(p_{n_1,j} + \sum_{i=n_1+1}^{k_1} p_{ij} \right). \tag{86}$$

Hence, with the help of (83), we obtain

$$I_{12} = \sum_{j=0}^{n_2-1} p_{n_1,j} + \sum_{i=n_1+1}^{k_1} \sum_{j=n_2}^{k_2} \left(\frac{a_1}{n_1} p_{i-1,j} - p_{ij} \right). \tag{87}$$

From (23) and (27), with the help of (11), (15), and (98), it is found that

$$\begin{aligned} I_{12} = & \sum_{r=0}^{k_1} b_r s_{n_1}(\kappa_r, a_1) s_{n_2-1}(1 - \kappa_r, a_2) \theta_{n_1}(\kappa_r) \\ & + \sum_{\ell=1}^{n_2} c_\ell s_{n_1}(\eta_\ell, a_1) s_{n_2-1}(1 - \eta_\ell, a_2) + \sum_{r=0}^{k_1} b_r s_{n_1}(\kappa_r, a_1) s_{n_2}(-\kappa_r, a_2) \\ & \cdot \left\{ \frac{a_1}{n_1} [\Omega_{q_1}(\kappa_r) - 1] - \Omega_{q_1-1}(\kappa_r) \right\} \frac{\Psi_{q_2}(\kappa_r)}{\phi_{n_2}(\kappa_r)}. \end{aligned} \tag{88}$$

Since $R_{12} = 0$ and $\lambda_1 = a_1 \mu$, the first relationship in (52) becomes

$$a_1(1 - L_1 - I_{12}) = X_1. \tag{89}$$

In view of (53), (64), and (88), this relationship provides a useful numerical check.

The mean departure rate from the primary queue to the primary servers is

$$R_{11} = n_1 \mu \sum_{i=n_1+1}^{k_1} \sum_{j=0}^{k_2} p_{ij}. \tag{90}$$

From (42) and (83), it follows that $R_{11} = \lambda_1 Q_1$, as expected. Finally, the average number of demands in the primary queue is

$$V_1 = \sum_{i=n_1+1}^{k_1} \sum_{j=0}^{k_2} (i - n_1) p_{ij}. \tag{91}$$

Hence, with the help of (11), (15), (23), (83), (113), and (115), it is found that

$$V_1 = b_0 s_{n_1}(0, a_1) s_{n_2}(0, a_2) \left(\frac{n_1}{a_1}\right)^{q_1} \Lambda_{q_1}\left(\frac{a_1}{n_1}\right) \frac{\Psi_{q_2}(0)}{\phi_{n_2}(0)}$$
$$+ \frac{a_1}{n_1} \sum_{r=1}^{k_1} b_r s_{n_1}(\kappa_r, a_1) s_{n_2}(-\kappa_r, a_2) \frac{\Psi_{q_2}(\kappa_r)}{\phi_{n_2}(\kappa_r)}$$
$$\cdot \left\{ \Omega_{q_1}(\kappa_r) - (q_1 + 1) + \frac{a_1}{\kappa_r} \left[\Omega_{q_1}(\kappa_r) - 1 - \frac{n_1}{a_1} \Omega_{q_1-1}(\kappa_r) \right] \right\}, \quad (92)$$

where $\Lambda_q(\xi)$ is as defined in (61).

APPENDIX

We define $s_i(\lambda, a)$ by the recurrence relation

$$(a + i + \lambda) s_i(\lambda, a) = a(1 - \delta_{i0}) s_{i-1}(\lambda, a) + (i + 1) s_{i+1}(\lambda, a);$$
$$s_0(\lambda, a) = 1, \quad (93)$$

for $i = 0, 1, \cdots$. Thus $s_n(\lambda, a)$ is a polynomial of degree n in both λ and a, and it may be related to a Poisson-Charlier polynomial.[4,5] However, we will give here the properties of $s_n(\lambda, a)$ which we will need. An explicit formula is[1]

$$s_i(\lambda, a) = \sum_{r=0}^{i} \frac{(\lambda)_r a^{i-r}}{r!(i-r)!}, \quad (94)$$

where

$$(\lambda)_0 = 1, \quad (\lambda)_r = \lambda(\lambda + 1) \cdots (\lambda + r - 1), \quad r = 1, 2, \cdots . \quad (95)$$

It was also shown[1] that

$$(i + 1) s_{i+1}(\lambda, a) = a s_i(\lambda, a) + \lambda s_i(\lambda + 1, a) \quad (96)$$

and

$$s_i(\lambda, a) = s_i(\lambda + 1, a) - (1 - \delta_{i0}) s_{i-1}(\lambda + 1, a). \quad (97)$$

From (97), it follows that

$$\sum_{i=0}^{n} s_i(\lambda, a) = s_n(\lambda + 1, a), \quad (98)$$

and, from (96) and (98), we deduce that

$$\sum_{i=0}^{n} (n - i) s_i(\lambda, a) = (1 - \delta_{n0}) s_{n-1}(\lambda + 2, a). \quad (99)$$

We now turn our attention to the Chebyshev polynomials of the second kind,[3] $U_\ell(x)$. They may be defined by the recurrence relation

$$2x U_\ell(x) = U_{\ell+1}(x) + U_{\ell-1}(x); \quad U_{-1}(x) \equiv 0, \quad U_0(x) \equiv 1, \quad (100)$$

for $\ell = 0, 1, \cdots$. From (10) and (100), it follows that

$$(a_2 + n_2 - \kappa)\Psi_\ell(\kappa) = a_2\Psi_{\ell+1}(\kappa) + n_2\Psi_{\ell-1}(\kappa);$$
$$\Psi_{-1}(\kappa) \equiv 0, \quad \Psi_0(\kappa) \equiv 1. \tag{101}$$

From (11) and (101), we deduce that

$$[a_2(1 - \delta_{jk_2}) + n_2 - \kappa]\phi_j(\kappa) = a_2\phi_{j-1}(\kappa) + n_2(1 - \delta_{jk_2})\phi_{j+1}(\kappa), \tag{102}$$

for $j \leq k_2$. Since[3]

$$U_\ell\left[\frac{1}{2}\left(\xi + \frac{1}{\xi}\right)\right] = \sum_{r=0}^{\ell} \xi^{2r-\ell}, \tag{103}$$

it follows that

$$\Psi_\ell(0) = \left(\frac{n_2}{a_2}\right)^{\ell/2} U_\ell\left(\frac{a_2 + n_2}{2\sqrt{a_2 n_2}}\right) = \sum_{r=0}^{\ell} \left(\frac{n_2}{a_2}\right)^r. \tag{104}$$

Hence, from (11), we have

$$\phi_j(0) = \left(\frac{n_2}{a_2}\right)^{k_2 - j}. \tag{105}$$

Now[3]

$$U_\ell(x) = \frac{[(x + \sqrt{x^2 - 1})^{\ell+1} - (x - \sqrt{x^2 - 1})^{\ell+1}]}{2\sqrt{x^2 - 1}}, \quad x \neq 1. \tag{106}$$

Hence, from (10), we obtain

$$\Psi_\ell(\kappa_r) = \left(\frac{n_2}{a_2}\right)^{\ell/2} \frac{[(\epsilon_r + \sqrt{\epsilon_r^2 - 1})^{\ell+1} - (\epsilon_r - \sqrt{\epsilon_r^2 - 1})^{\ell+1}]}{2\sqrt{\epsilon_r^2 - 1}}, \tag{107}$$

for $r = 1, \cdots, k_1$, where ϵ_r is given by (20). We note that $\epsilon_r > 1$ for $r = 1, \cdots, k_1$, since $\kappa_r < 0$ for $r = 1, \cdots, k_1$. We take the positive square root of $\epsilon_r^2 - 1$ in (107). From (11), it follows that

$$\begin{aligned}\phi_j(\kappa_r) = {} & \frac{(\epsilon_r + \sqrt{\epsilon_r^2 - 1} - \sqrt{a_2/n_2})}{2\sqrt{\epsilon_r^2 - 1}} \left[\sqrt{\frac{n_2}{a_2}}\,(\epsilon_r + \sqrt{\epsilon_r^2 - 1})\right]^{k_2 - j} \\ & - \frac{(\epsilon_r - \sqrt{\epsilon_r^2 - 1} - \sqrt{a_2/n_2})}{2\sqrt{\epsilon_r^2 - 1}} \left[\sqrt{\frac{n_2}{a_2}}\,(\epsilon_r - \sqrt{\epsilon_r^2 - 1})\right]^{k_2 - j},\end{aligned} \tag{108}$$

for $r = 1, \cdots, k_1$. Since $\kappa_r < 0$ for $r = 1, \cdots, k_1$, we deduce from (20) that

$$\epsilon_r + \sqrt{\epsilon_r^2 - 1} > \epsilon_0 + \sqrt{\epsilon_0^2 - 1} = \max\left(\sqrt{\frac{a_2}{n_2}}, \sqrt{\frac{n_2}{a_2}}\right) \geq \sqrt{\frac{a_2}{n_2}} \tag{109}$$

for $r = 1, \cdots, k_1$.

Next, from (14) and (100), it follows tbat

$$(a_1 + n_1 + \kappa)\Omega_\ell(\kappa) = a_1\Omega_{\ell+1}(\kappa) + n_1\Omega_{\ell-1}(\kappa);$$

$$\Omega_{-1}(\kappa) \equiv 0, \quad \Omega_0(\kappa) \equiv 1. \tag{110}$$

From (15) and (110), we deduce that

$$[a_1(1 - \delta_{ik_1}) + n_1 + \kappa]\theta_i(\kappa) = a_1\theta_{i-1}(\kappa) + n_1(1 - \delta_{ik_1})\theta_{i+1}(\kappa), \tag{111}$$

for $i \le k_1$. Also, from (14) and (103), we obtain

$$\Omega_\ell(0) = \left(\frac{n_1}{a_1}\right)^{\ell/2} U_\ell\left(\frac{a_1 + n_1}{2\sqrt{a_1 n_1}}\right) = \sum_{r=0}^{\ell} \left(\frac{n_1}{a_1}\right)^r. \tag{112}$$

Hence, from (15), we have

$$\theta_i(0) = \left(\frac{n_1}{a_1}\right)^{k_1 - i}. \tag{113}$$

Since $k_1 = n_1 + q_1$, it follows that

$$\sum_{i=n_1}^{k_1} (i - n_1)\theta_i(0) = \left(\frac{n_1}{a_1}\right)^{q_1} \Lambda_{q_1}\left(\frac{a_1}{n_1}\right), \tag{114}$$

where $\Lambda_q(\xi)$ is as defined in (61). Finally, if we multiply (111) by $(i - n_1)$ and sum on i, we obtain, with the help of (15),

$$\kappa \sum_{i=n_1}^{k_1} (i - n_1)\theta_i(\kappa) = a_1[\Omega_{q_1}(\kappa) - 1] - n_1\Omega_{q_1-1}(\kappa). \tag{115}$$

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